Math 2450: Limits and Continuity

What is a limit of a function of two variables? Before considering a function of two variables, let's think back to Calculus I, when we learned how to take limits of one variable functions. Recall that:

$$\lim_{x\to a} f(x) = L \quad \text{if} \quad \lim_{x^+\to a} f(x) = L \quad \text{AND} \quad \lim_{x^-\to a} f(x) = L$$

In other words, $\lim_{x\to a} f(x) = L$ provided $f(x)$ approaches L as the value of x approaches a from the left and right hand sides.

Similarly, to show a limit of a two variable function exists, the function $f(x, y)$ must be approaching the same value, L, regardless of the path we take toward the point (x_0, y_0).

Example 1. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2 - 2xy + y^2}{x - y}$

What if the limit does not exist? Remember that $f(x, y)$ must approach the value L no matter the path taken for $(x, y) \to (x_0, y_0)$. That means that (x, y) can approach (x_0, y_0) along **any** curve that passes through (x_0, y_0). Therefore, if there exists any path (or curve) within the domain of f where $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y) \neq L$, then the limit does not exist.

Example 2. Determine whether or not the limit exists. If the limit does exist, state the value of the limit.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4}$$

Note: Once we choose a path and have a limit in 1 variable, we may solve the limit as in Calc I (e.g. reduce / direct substitution / L'Hospital).

$\star$**Note:** We also could have checked the pathways along $y = x^2$, $x = y^2$, $y = x^3$, and so on. There is no general rule for how many pathways to check before declaring a limit does not exist and, therefore, it may take a little work to show.

Limits Rules
Similar to limits of one variable functions, limits of two variable functions satisfy the scalar multiple, sum, product, and quotient rule.

What is continuity? The function $f(x, y)$ is **continuous** at the point (x_0, y_0) if and only if

1. $f(x_0, y_0)$ is defined
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$

Recall Example 1, when we used direct substitution and got the undefined expression $\frac{0}{0}$. According to the definition of continuity, $f(x, y) = \frac{x^2-2xy+y^2}{x-y}$ is discontinuous at $(0, 0)$. There is a "hole" in the surface of $f(x, y)$ at $(0, 0)$.

Practice Problems
Determine if the limits exist or not. If they do exist, give the value of the limit. Are the functions continuous at $(0, 0)$?

1. $\lim_{(x,y)\to(0,0)} \dfrac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$ [**Solution:** $\frac{5}{2}$, Continuous at $(0, 0)$]
2. $\lim_{(x,y)\to(0,0)} \dfrac{x^2 + 7xy + 12y^2}{x + 3y}$ [**Solution:** 0, Discontinuous at $(0, 0)$ since $f(0, 0)$ is undefined]
3. $\lim_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^4 + y^2}$

   [**Solution:** Does not exist (check along $y = x$ and $y = x^2$), Discontinuous at $(0, 0)$]